On polynomials of optimal shape and the number of period annuli

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Summary. We construct polynomials of even degree which have an optimal in some sense distribution of local maxima. Any of such polynomial gives rise to a conservative equation $x'' + g(x) = 0$, which possesses multiple period annuli.

MSC: 34B15, 34C25

1 Introduction

Let $G(x)$ be a polynomial of even degree with branches downwards which has multiple local maxima. Consider differential equation

$$x'' + g(x) = 0, \tag{1}$$

where $g(x) = G'(x)$ is an odd degree polynomial with simple zeros.

The equivalent differential system

$$x' = y, \quad y' = -g(x) \tag{2}$$

has critical points at $(p_i, 0)$, where p_i are simple zeros of $g(x)$. Recall that a critical point O of (2) is a center if it has a punctured neighborhood covered with nontrivial cycles.

Definition 1.1 ([3]) *A* **central region** *is the largest connected region covered with cycles surrounding O.*

Definition 1.2 ([3]) *A* **period annulus** *is every connected region covered with nontrivial concentric cycles.*

Definition 1.3 *We will call a period annulus associated with a central region* **a trivial period annulus**. *Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center.*

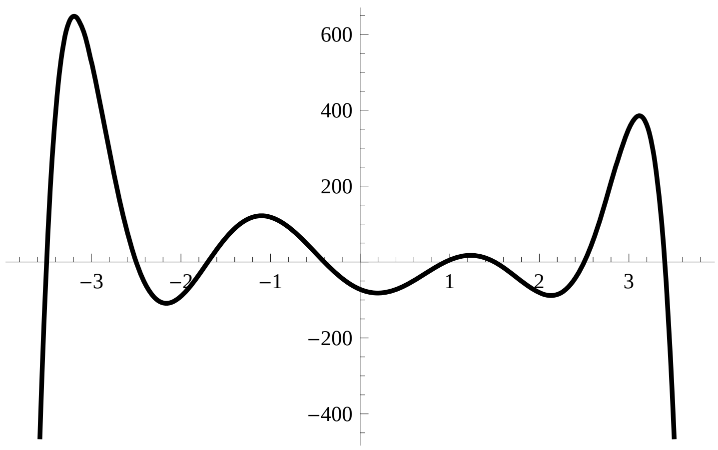

Fig. 1.1. Polynomial of 8-th degree with optimal distribution of maxima

Definition 1.4 *Respectively a period annulus enclosing several (more than one) critical points will be called a* **nontrivial period annulus**.

This equation is known [1] to have multiple *nontrivial period annuli* if the function $G(x)$ has multiple *regular pairs* of maxima (Definition 3.1).

We consider the task of defining the maximal number of nontrivial period annuli for equation (1).

A. We suppose that $g(x)$ is an odd degree polynomial with the negative principle term with simple zeros only. A zero z is called simple if $g(z) = 0$ and $g'(z) \neq 0$.

Our study is conducted in several stages. First functions g are considered with the primitive functions G that have multiple non-equal local maxima. It is shown by induction that the maximal possible number of non-neighboring local maxima is $n - 2$ where n is the number of local maxima of the function $G(x)$. The structure of the graph of $G(x)$ which yield the optimal number of period annuli is called *optimal.*

In the second stage the scheme of construction of $G(x)$ with the optimal structure is discussed.

2 Nontrivial period annuli

The result below provides us with a criterium for existence of a nontrivial period annulus.

Theorem 2.1 ([1]) *Let the condition* ***(A)*** *hold. Suppose that M_1 and M_2 ($M_1 < M_2$) are non-neighboring points of maximum of the function $G(x)$. Suppose that any other local maximum of $G(x)$ in the interval (M_1, M_2) is (strictly) less than min$\{G(M_1); G(M_2)\}$.*

Then there exists a nontrivial period annulus associated with a pair (M_1, M_2).

It is evident that if $G(x)$ has m pairs of non-neighboring points of maxima then m nontrivial period annuli exist.

Consider for example equation (1), where

$$g(x) = -x(x+3)(x+2.2)(x+1.9)(x+0.8)(x-0.3)(x-1.5)(x-2.3)(x-2.9). \quad (3)$$

The equivalent system has alternating "saddles" and "centers" and the graph of $G(x)$ is depicted in Fig. 2.1.

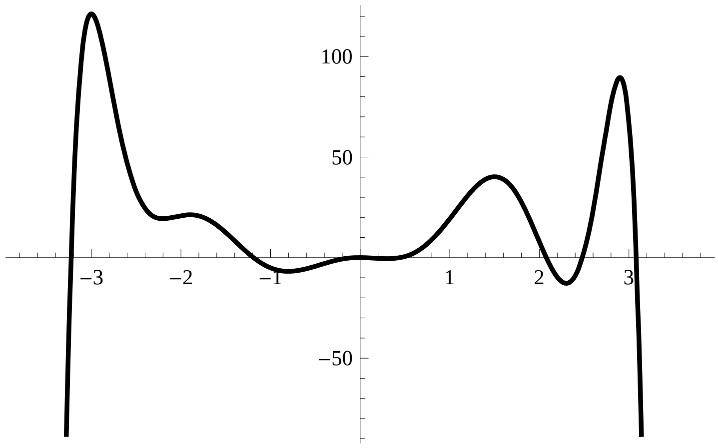

Fig. 2.1.

There are three pairs of non-neighbouring points of maxima and three nontrivial period annuli exist, which are depicted in Fig. 2.2.

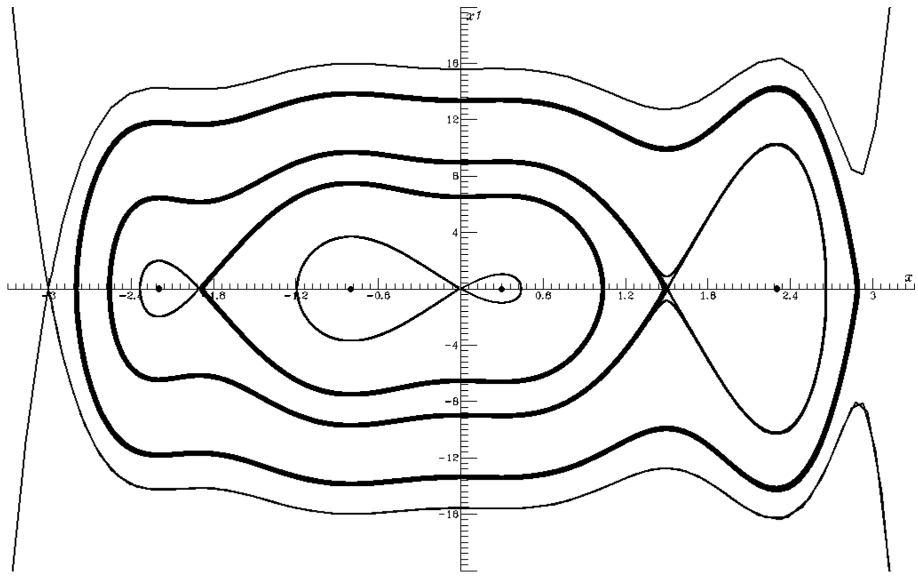

Fig. 2.2.Phase plane for equation $x'' + g(x) = 0$ where $g(x)$ is given by (3)

3 Polynomials

Consider the function $G(x)$. Points of local maxima x_i and x_j of $G(x)$ are non-neighboring if the interval (x_i, x_j) contains at least one point of local maximum of $G(x)$. Let $G(x)$ be a continuous function on an interval $I \subset R$, and let x_i and x_j be points of local maxima of $G(x)$, which belong to I, $x_i < x_j,\ i < j$.

Definition 3.1 *Two non-neighboring points of maxima $x_i < x_j$ of $G(x)$ will be called a* **regular pair** *if $G(x) < \min\{G(x_i), G(x_j)\}$ at any other point of maximum lying in the interval (x_i, x_j).*

Theorem 3.1 *Let $G(x)$ be a primitive (anti-derivative) of the function $g(x)$ satisfying the condition* **A.** *Let n be the number of points of maxima of $G(x)$.*

The possible maximal number of **regular pairs** *is $n - 2$.*

Proof. By induction. Let $x_1, x_2, ..., x_n$ be a set of successive points of maxima of $G(x),\ x_1 < x_2 < ... < x_n$.

1) Let $n = 3$. The following combinations are possible at three points of maxima.

$$\begin{array}{ll} a)\ \ G(x_1) \geq G(x_2) \geq G(x_3) & b)\ G(x_2) < G(x_1),\ G(x_2) < G(x_3) \\ c)\ \ G(x_1) \leq G(x_2) \leq G(x_3) & d)\ G(x_2) \geq G(x_1),\ G(x_2) \geq G(x_3) \end{array}$$

Only in the case $b)$ there exist *a regular pair.* In this case therefore the maximal number of *regular pairs* is 1.

2) Suppose that for any sequence of $n > 3$ ordered points of maxima of $G(x)$ the maximal number of *regular pairs* is $n-2$. Without loss of generality add to the right one more point of maximum of the function $G(x)$. We get a sequence of $n+1$ consecutive points of maximum $x_1, x_2, \ldots, x_n, x_{n+1},\ x_1 < x_2 < \ldots < x_n < x_{n+1}$. Let us prove that the maximal number of *regular pairs* is $n - 1$. For this consider the following possible variants.

a) A couple x_1, x_n is *a regular pair.* If $G(x_1) > G(x_n)$ and $G(x_{n+1}) > G(x_n)$, then, besides of the *regular pairs* in the interval $[x_1, x_n]$, only one new *regular pair* can appear, namely x_1, x_{n+1}. Then the maximal number of *regular pairs* which can be composed of the points $x_1, x_2, \ldots, x_n, x_{n+1}$, is not greater than $(n-2)+1 = n-1$. If $G(x_1) \leq G(x_n)$ or $G(x_{n+1}) \leq G(x_n)$, then the additional *regular pair* does not appear. In a particular case $G(x_2) < G(x_3) < ... < G(x_n) < G(x_{n+1})$ and $G(x_1) > G(x_n)$ the following *regular pairs* emerge, namely, x_1 and x_3, x_1 and x_4, $\ldots$, x_1 and x_n, x_1 and x_{n+1}, totally $n - 1$ pairs.

b) Let x_i and x_j be a *regular pair*, $1 \leq i < j \leq n$, and there is no a *regular pair* x_p, x_q such that $1 \leq p \leq i < j \leq q \leq n$. This means that $G(x_i) \geq G(x_p)$ for any $p = 1, ..., i-1$ and $G(x_j) \geq G(x_q)$ for any $q = j+1, ..., n$. Let us mention that if such a pair x_i, x_j does not exist then the function $G(x)$ does not have *regular pairs* at all. The interval $[x_1, x_i]$ contains i points of maximum of $G(x),\ i < n$, and hence the number of *regular pairs* in this interval does not exceed $i - 2$. The interval $[x_i, x_j]$ contains $j - (i-1) = j - i + 1$ points of maximum of $G(x),\ j - i + 1 \leq n$, and hence the number of *regular pairs* in this interval does not exceed $j - i + 1 - 2 = j - i - 1$. If $G(x_{n+1}) > G(x_j)$ and $G(x_i) > G(x_j)$, then, besides of *regular pairs* in $[x_j, x_{n+1}]$, there exists one more *regular pair* x_i, x_{n+1}. If $G(x_{n+1}) \leq G(x_j)$ or $G(x_i) \leq G(x_j)$, then this additional *regular pair* does not appear.

The interval $[x_j, x_{n+1}]$ contains $(n+1)-(j-1) = n-j+2$ points of maximum of $G(x)$, $n-j+2 \le n$. The number of *regular pairs* in this interval is not greater than $(n-j+2)-2+1 = n-j+1$. Note that in view of the restrictions on the pair x_i, x_j there are not *regular pairs* x_p, x_q such that

$$\begin{cases} 1 \le p < i, \\ i < q \le n \end{cases} \text{or} \begin{cases} 1 \le p < j, \\ j < q \le n. \end{cases}$$

Therefore the number of *regular pairs* in the interval $[x_1, x_{n+1}]$ is not greater than $(i-2)+(j-i-1)+(n-j+1) = n-2 < n-1$. It follows from the above argument that the maximal possible number of *regular pairs* in the interval $[x_1, x_{n+1}]$, which contains $n+1$ points of maximum of $G(x)$, is equal to $n-1$. $\square$

4 The existence of optimal polynomials

Theorem 4.1 *Given number n a polynomial $g(x)$ satisfying the condition* **A** *can be constructed such that the primitive function $G(x)$ has exactly n points of maximum and the number of* regular pairs *is exactly $n-2$.*

Proof of the theorem. Consider the polynomial

$$G(x) = -(x+\frac{1}{2})(x-\frac{1}{2})(x+\frac{3}{2})(x-\frac{3}{2})(x+\frac{5}{2})(x-\frac{5}{2})(x+\frac{7}{2})(x-\frac{7}{2}). \qquad (4)$$

It is an even function with the graph depicted in Fig. 4.1.

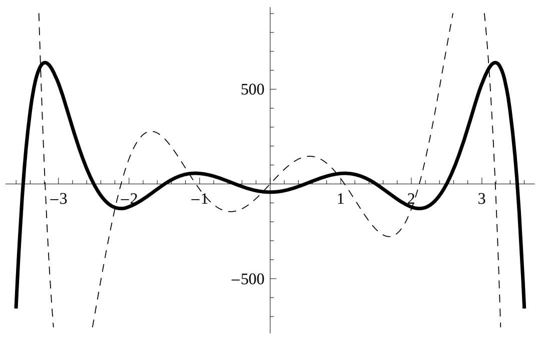

Fig. 4.1. $G(x)$ (solid) and $G'(x) = g(x)$ (dashed)

Consider now the polynomial

$$G_\varepsilon(x) = -(x+\frac{1}{2}+\varepsilon)(x-\frac{1}{2})(x+\frac{3}{2})(x-\frac{3}{2})(x+\frac{5}{2})(x-\frac{5}{2})(x+\frac{7}{2})(x-\frac{7}{2}), \qquad (5)$$

where $\varepsilon > 0$ is small enough. The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Fig. 4.2.

Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the right of $x=0$ m_1^+, m_2^+. Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the left of $x=0$ m_1^-, m_2^-. One has for $G(x)$ that $m_1^+ = m_1^- < m_2^- = m_2^+$. One has for $G_\varepsilon(x)$ that $m_1^+ < m_1^- < m_2^+ < m_2^-$. Then there are two regular pairs (respectively, m_1^- and m_2^+, m_2^+ and m_2^-).

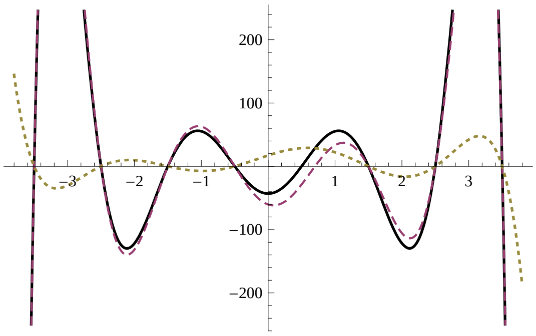

Fig. 4.2. $G(x)$ - solid, $G_\varepsilon(x)$ - dashed, $G(x) - G_\varepsilon(x)$-dotted

For arbitrary even n the polynomial

$$G_\varepsilon(x) = -(x + \frac{1}{2})(x - \frac{1}{2})(x + \frac{3}{2})(x - \frac{3}{2}) \dots (x + \frac{2n-1}{2})(x - \frac{2n-1}{2}), \tag{6}$$

is to be considered where the maximal values m_1^+, m_2^+, ..., $m_{n/2}^+$ to the right of $x = 0$ form ascending sequence, and, respectively, the maximal values m_1^-, m_2^-, ..., $m_{n/2}^-$ to the left of $x = 0$ form descending sequence. For a slightly modified polynomial

$$G_\varepsilon(x) = -(x + \frac{1}{2} + \varepsilon)(x - \frac{1}{2})(x + \frac{3}{2})(x - \frac{3}{2}) \dots (x + \frac{2n-1}{2})(x - \frac{2n-1}{2}), \tag{7}$$

the maximal values are arranged as

$$m_1^+ < m_1^- < m_2^+ < m_2^- < \dots < m_{n/2}^+ < m_{n/2}^-.$$

Therefore there exist exactly $n-2$ *regular pairs* and consequently $n-2$ nontrivial period annuli in the differential equation (1).

If n is odd then the polynomial

$$G(x) = -x^2(x-1)(x+1)(x-2)(x+2)\dots(x-(n-1))(x+(n-1)) \tag{8}$$

with n local maxima is to be considered. The maxima are descending for $x < 0$ and ascending if $x > 0$. The polynomial with three local maxima is depicted in Fig. 4.3.

The slightly modified polynomial

$$G(x) = -x^2(x-1-\varepsilon)(x+1)(x-2)(x+2)\dots(x-(n-1))(x+(n-1)) \tag{9}$$

has maxima which are not equal and are arranged in an optimal way in order to produce the maximal $(n-2)$ *regular pairs.*

The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Fig. 4.4.

References

[1] S. Atslega, F. Sadyrbaev. Multiple solutions of the second order nonlinear Neumann BVP. Dynamics of Continuous, Discrete and Impulsive Systems (Series A). DCDIS A Supplement dedicated to the 6th Intern. Conf. Diff. Equations and Dynam. Systems held in Baltimore, U.S.A., May 22- 26, Watam Press, 2009, 100-103.

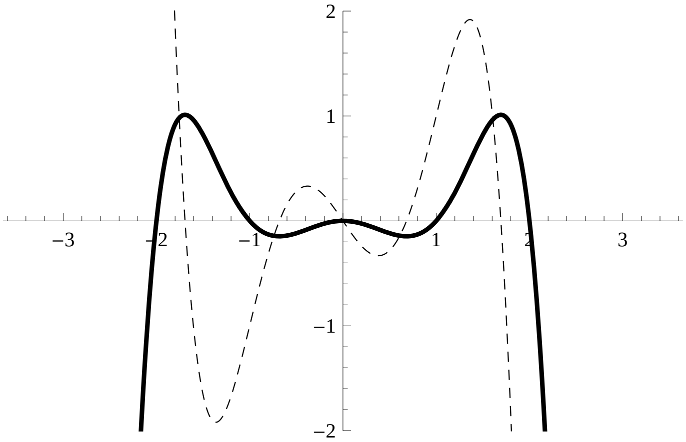

Fig. 4.3. $G(x)$ (solid) and $G'(x)$ (dashed)

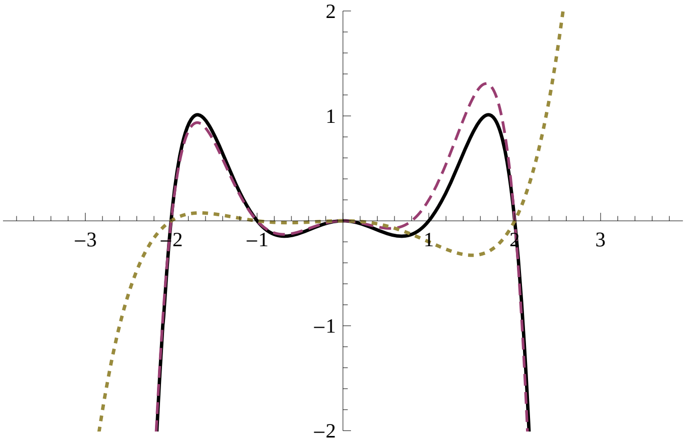

Fig. 4.4. $G(x)$ - solid, $G_\varepsilon(x)$ - dashed, $G(x) - G_\varepsilon(x)$-dotted

[2] S. Ogorodnikova, F. Sadyrbaev, Multiple solutions of nonlinear boundary value problems with oscillatory solutions. Mathematical Modelling and Analysis, 11 (2006), N 4, 413 – 426.

[3] M. Sabatini, Liénard limit cycles enclosing period annuli, or enclosed by period annuli. *Rocky Mount. J. Math.*, **35** (2005), N 1, 253 - 266.

Е. Козьмина, Ф. Садырбаев. О полиномах оптимального профиля и числе периодических колец.

Аннотация. Указывается максимальное число нетривиальных периодических колец для уравнения $x'' + g(x) = 0$, где $g(x)$ полином с простыми нулями.

УДК 517.927

Je. Kozmina, F. Sadirbajevs. Par optimāla profila polinomiem un periodisko gredzenu skaitu

Anotācija. Tiek atrasts maksimāls netriviālo periodisko gredzenu skaits vienādojumam $x'' + g(x) = 0$, kur $g(x)$ ir polinoms ar vienkāršam saknēm.

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