The Upper and Lower Functions Method for One-dimensional Φ-Laplacians

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Summary. We consider equation $(\phi(t,x,x'))' = f(t,x,x')$ with the boundary conditions $x(a) = A$, $x(b) = B$. Upper and lower functions are defined in a relatively general manner. It is shown that this definition is best possible in some sense. In the second part generalizations of Bernstein - Nagumo type conditions are developed for the case under consideration.

MSC: 34B15, 34C25

1 Introduction

The classical result (which can be found, for example, in [1, chapter 1, theorem 1.5.1], or [13, chapter 3, §1, theorem 1]) for the boundary value problem

$$x'' = f(t,x), \quad t \in I := [a,b], \quad f \in C(I \times I\!R, I\!R)$$

$$x(a) = A, \quad x(b) = B \tag{1}$$

states that a solution to this problem exists if there are two functions $\alpha, \beta \in C^2(I, I\!R)$ with the properties:
(C0) $\alpha''(t) \geq f(t,\alpha(t)), \quad \beta''(t) \leq f(t,\beta(t))$;
(C1) $\alpha(t) \leq \beta(t) \quad \forall t \in I$;
(C2) $\alpha(a) \leq A \leq \beta(a), \quad \alpha(b) \leq B \leq \beta(b)$.
Moreover, the existence of a solution x, which satisfies the estimate $\alpha \leq x \leq \beta$, is assured. In the case of a more general equation

$$x'' = f(t,x,x') \tag{2}$$

with a continuous right side the so called Bernstein - Nagumo type conditions should be imposed additionally in order to guarantee solvability of the problem (2), (1). If the ratio $\frac{f(t,x,y)}{y^2}$ is bounded for y large in modulus uniformly in (t,x) then it is used to say that the

Bernstein condition is fulfilled which (together with the existence of α and β) is enough for existence of a solution to the problem (2), (1). For discussion on the method of upper and lower functions and Bernstein - Nagumo type conditions one may consult the books [1], [10, Ch. XII], [13, Ch. 3].

In this paper we study equation

$$(\phi(t,x,x'))' = f(t,x,x') \tag{3}$$

with $\phi \in C(I \times \mathbb{R}^2, \mathbb{R})$ strictly increasing in x' function and f being a Carathéodory function. We mention the following papers devoted to the same subject, [7], [5], [18], [9]. Evidently, (3) is a generalization of equation (2). Various areas, where equations of the form (3) arise, are indicated in references above. Let us mention also that the Euler - Lagrange equation for the functional $J(x) = \int_a^b L(t,x,x')\,dt$ also is of the form (3).

Our goal in this paper is two-fold. First, we define upper and lower functions for equation (3) and show that this definition is final in some sense. Roughly speaking, we show that the class of lower functions according to our definition coincides with the class of all Lipschitz functions possessing the property of *the one-sided solvability from above* with respect to equation (3). Analogously the class of upper functions coincides with the class of all Lipschitz functions possessing the property of *the one-sided solvability from below.* Precise definitions and statements of results are given in the Section 2. Analogous results for the second order equations in the form (2) where obtained in the book [14, Ch. 1]. Second, we develope a set of Bernstein - Nagumo type conditions which take into account the specific form of equation (3). The Section 3 is devoted to Bernstein - Nagumo type conditions and contains discussion on the subject.

2 Upper and lower functions

Suppose that ϕ in (3) is as described in the previous section and f is a Carathéodory function, that is, (i) $f(\cdot,x,y)$ is measurable in I for $(x,y) \in \mathbb{R}^2$ fixed; (ii) $f(t,\cdot,\cdot)$ is continuous in $\mathbb{R}^2$ for a.e. $t \in I$; (iii) for any compact set $P \subset \mathbb{R}^2$ there exists a function $g \in L_1(I,\mathbb{R})$ such that $|f(t,x,y)| \le g(t)$ holds in $I \times P$.

Definition 1. Let $c \in [a,b)$ and $d \in (c,b]$. A function $x \in C^1([c,d],\mathbb{R})$ is a solution of the equation (3), if $\phi(t,x(t),x'(t)) : [c,d] \to \mathbb{R}$ is an absolutely continuous function and equation (3) is satisfied a.e. in $[c,d]$.

Definition 2. Functions $\alpha, \beta \in \mathrm{Lip}(I,\mathbb{R})$ will be called *lower and upper functions* for equation (3) if for any points $t_1 \in (a,b)$ and $t_2 \in (t_1,b)$, in which the first order derivatives exist, the inequalities

$$\begin{aligned} \phi(t_2,\alpha(t_2),\alpha'(t_2)) - \phi(t_1,\alpha(t_1),\alpha'(t_1)) &\ge \int_{t_1}^{t_2} f(s,\alpha(s),\alpha'(s))\,ds \\ \phi(t_2,\beta(t_2),\beta'(t_2)) - \phi(t_1,\beta(t_1),\beta'(t_1)) &\le \int_{t_1}^{t_2} f(s,\beta(s),\beta'(s))\,ds \end{aligned} \tag{4}$$

hold respectively.

The sets of all lower and all upper functions will be denoted by $A(I,\mathbb{R})$ and $B(I,\mathbb{R})$ respectively.

Lemma 2.1 *If $\alpha \in A(I, I\!R)$ and $\beta \in B(I, I\!R)$, then the left derivatives $\alpha_l'(t), \beta_l'(t)$ exist for any $t \in (a, b]$, the right derivatives $\alpha_r'(t), \beta_r'(t)$ exist for any $t \in [a, b)$, the inequalities $\alpha_l'(t) \le \alpha_r'(t)$ and $\beta_l'(t) \ge \beta_r'(t)$ hold in (a, b), the following relations*

$$\begin{array}{ll} \lim_{\tau\to t+} \alpha_l'(\tau) = \lim_{\tau\to t+} \alpha_r'(\tau) = \alpha_r'(t), & t \in [a, b) \\ \lim_{\tau\to t-} \alpha_l'(\tau) = \lim_{\tau\to t-} \alpha_r'(\tau) = \alpha_l'(t), & t \in (a, b] \\ \lim_{\tau\to t+} \beta_l'(\tau) = \lim_{\tau\to t+} \beta_r'(\tau) = \beta_r'(t), & t \in [a, b) \\ \lim_{\tau\to t-} \beta_l'(\tau) = \lim_{\tau\to t-} \beta_r'(\tau) = \beta_l'(t), & t \in (a, b] \end{array} \tag{5}$$

are valid and for any $t_1 \in [a, b)$ and $t_2 \in (t_1, b]$ the inequalities

$$\begin{array}{l} \phi(t_2, \alpha(t_2), \alpha_l'(t_2)) - \phi(t_1, \alpha(t_1), \alpha_r'(t_1)) \ge \displaystyle\int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))\, ds \\ \phi(t_2, \beta(t_2), \beta_l'(t_2)) - \phi(t_1, \beta(t_1), \beta_r'(t_1)) \le \displaystyle\int_{t_1}^{t_2} f(s, \beta(s), \beta'(s))\, ds \end{array} \tag{6}$$

are satisfied.

Proof. Let T_α be the set of those $\tau \in (a, b)$, for which the derivative $\alpha'(\tau)$ exists. We will show first that the limit

$$\lim_{\tau\to t+} \alpha'(\tau) =: \alpha_r(t), \quad t \in [a, b) \tag{7}$$

exists, $\tau \in T_\alpha$. Suppose that

$$\liminf_{\tau\to t+} \alpha'(\tau) < c_1 < c_2 < \limsup_{\tau\to t+} \alpha'(\tau). \tag{8}$$

Pick $t_1, t_2 \in T_\alpha$ such that $t < t_1 < t_2$, $\alpha'(t_1) > c_2$ and $\alpha'(t_2) < c_1$. The first of the inequalities (4) and monotonicity of ϕ together imply that

$$\begin{array}{rcl} \displaystyle\int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))\, ds & \le & \phi(t_2, \alpha(t_2), \alpha'(t_2)) - \phi(t_1, \alpha(t_1), \alpha'(t_1)) \\ & < & \phi(t_2, \alpha(t_2), c_1) - \phi(t_1, \alpha(t_1), c_2). \end{array} \tag{9}$$

On the other hand, if t_1 and t_2 are sufficiently close to t, then

$$\begin{array}{rcl} \displaystyle\int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))\, ds & > & \dfrac{\phi(t, \alpha(t), c_1) - \phi(t, \alpha(t), c_2)}{2} \\ & > & \phi(t_2, \alpha(t_2), c_1) - \phi(t_1, \alpha(t_1), c_2). \end{array} \tag{10}$$

The contradiction obtained means that the limit in (7) exists.

The existence of a limit in (7) and an absolute continuity of α together imply that the derivative $\alpha_r'(t) = \alpha_r(t)$ exists. The existence of $\alpha_l'(t)$ for any $t \in (a, b]$ can be shown analogously. If $\alpha_r'(t) < c_1 < c_2 < \alpha_l'(t)$ for $t \in (a, b)$ then for $t_1 < t < t_2$ the inequalities (9) and (10) hold and a contradiction follows also. Hence $\alpha_l'(t) \le \alpha_r'(t)$, $t \in (a, b)$. The relations

$$\lim_{\tau\to t+} \alpha'(\tau) = \alpha_r'(t), \quad t \in [a, b),$$
$$\lim_{\tau\to t-} \alpha'(\tau) = \alpha_l'(t), \quad t \in (a, b]$$

imply formulas (5) and (6) with respect to α. Proofs for β are similar.

Lemma 2.2 *If* $\alpha_1, \alpha_2 \in A(I, I\!R)$, *then* $\max\{\alpha_1, \alpha_2\} \in A(I, I\!R)$. *If* $\beta_1, \beta_2 \in B(I, I\!R)$, *then* $\min\{\beta_1, \beta_2\} \in B(I, I\!R)$.

Proof. It is clear that $\alpha = \max\{\alpha_1, \alpha_2\} \in \mathrm{Lip}(I, I\!R)$. Let us show that the condition (4) holds for α. Denote $J = \{t \in (a,b) : \alpha_1(t) < \alpha_2(t)\}$. Clearly J is a union of open intervals. In cases of $J = \emptyset$ or $J = (a,b)$ the condition (4) evidently holds. Consider the case of $J = (a_1, b_1)$. Let $t_1 \in (a, a_1)$ and $t_2 \in (a_1, b_1)$. Then

$$\phi(a_1, \alpha_1(a_1), \alpha'_{1l}(a_1)) - \phi(t_1, \alpha_1(t_1), \alpha'_1(t_1))$$

$$+\phi(t_2, \alpha_2(t_2), \alpha'_2(t_2)) - \phi(a_1, \alpha_2(a_1), \alpha'_{2r}(a_1))$$

$$\geq \int_{t_1}^{a_1} f(t, \alpha_1(t), \alpha'_1(t))\, dt + \int_{a_1}^{t_2} f(t, \alpha_2(t), \alpha'_2(t))\, dt$$

$$= \int_{t_1}^{t_2} f(t, \alpha(t), \alpha'(t))\, dt.$$

The inequality $\alpha'_{1l}(a_1) \leq \alpha'_{2r}(a_1)$ implies that

$$\phi(a_1, \alpha(a_1), \alpha'_{1l}(a_1)) - \phi(a_1, \alpha(a_1), \alpha'_{2r}(a_1)) \leq 0.$$

Hence the condition (4). Other cases of location of t_1 and t_2 can be treated similarly.

Consider the case of $J = (a_1, b_1) \cup \ldots \cup (a_n, b_n)$. Let $\alpha_3(t) = \alpha_1(t)$, $t \in I \setminus [a_1, b_1]$ and $\alpha_3(t) = \alpha_2(t)$, $t \in [a_1, b_1]$. It follows from the arguments above that the condition (4) is fulfilled for α_3. Let $\alpha_4(t) = \alpha_3(t)$, $t \in I \setminus [a_2, b_2]$ and $\alpha_4(t) = \alpha_2(t)$, $t \in [a_2, b_2]$. It is clear that the condition (4) is fulfilled for α_4. Proceeding in the same manner, one can prove that $\alpha := \alpha_{n+2}$ satisfies the condition (4).

If there are infinitely many intervals (a_i, b_i), a sequence of α_i $(i = 3, 4, \ldots)$ can be constructed as above. It is clear that

$$\phi(t_2, \alpha_i(t_2), \alpha'_i(t_2)) - \phi(t_1, \alpha_i(t_1), \alpha'_i(t_1)) \geq \int_{t_1}^{t_2} f(t, \alpha_i(t), \alpha'_i(t))\, dt.$$

The condition (4) follows from the inequality above by passing to the limit as $i \to \infty$.

Proof for $\beta = \min\{\beta_1, \beta_2\}$ is similar.

Definition 3. We say that *a one-sided local solvability from above* holds for a function $\alpha : I \to I\!R$, if for any $\tau \in I$ and $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that for any $t_1, t_2 \in I$, which satisfy the inequalities $\tau - \delta < t_1 < t_2 < \tau + \delta$, a solution $x : [t_1, t_2] \to I\!R$ of the equation (3) exists with the properties

$$x(t_1) = \alpha(t_1), \quad x(t_2) = \alpha(t_2), \quad \alpha(t) \leq x(t) \leq \alpha(t) + \varepsilon \quad \forall t \in [t_1, t_2].$$

Similarly, *a one-sided local solvability from below* holds for a function $\beta : I \to I\!R$, if for any $\tau \in I$ and $\varepsilon \in (0, +\infty)$ there exists $\delta \in (0, +\infty)$ such that for any $t_1, t_2 \in I$, which satisfy the inequalities $\tau - \delta < t_1 < t_2 < \tau + \delta$, a solution $x : [t_1, t_2] \to I\!R$ of the equation (3) exists with the properties

$$x(t_1) = \beta(t_1), \quad x(t_2) = \beta(t_2), \quad \beta(t) \geq x(t) \geq \beta(t) - \varepsilon \quad \forall t \in [t_1, t_2].$$

We prove below the following statement.

Theorem 2.1 *If* $\alpha \in \mathrm{Lip}(I, I\!R)$ *satisfies the one-sided local solvability condition from above then* $\alpha \in A(I, I\!R)$*. If* $\beta \in \mathrm{Lip}(I, I\!R)$ *satisfies the one-sided local solvability condition from below then* $\beta \in B(I, I\!R)$*.*

Proof. Construct a sequence of $\alpha_n \in A(I, I\!R)$ $(n = 1, 2, \ldots)$ in the following way. Let $t \in I$, $\varepsilon = 1$ and let δ_t be a respective constant mentioned in Definition 3. A system of intervals $(t - \delta_t, t + \delta_t)$ form a covering of the interval I. Consider a finite subcovering which will be denoted (a_i, b_i) $(i = 1, \ldots, m)$. We may assume without loss of generality that

$$a_1 < a < a_2 < b_1 < a_3 < b_2 < \ldots < a_m < b_{m-1} < b < b_m.$$

Pick $c_i \in (a_{i+1}, b_i)$ $(i = 1, \ldots, m-1)$ so that there exist the derivatives $\alpha'(c_i)$. We may assume that $c_{i+1} - c_i < 1$ $(i = 0, \ldots, m-1)$, where $c_0 = a$ and $c_m = b$. Otherwise new partition points can be added. On any interval $[c_{i-1}, c_i]$ $(i = 1, \ldots, m)$ there exists a solution $x_i : [c_{i-1}, c_i] \to I\!R$ of equation (3) such that $x_i(c_{i-1}) = \alpha(c_{i-1})$, $x_i(c_i) = \alpha(c_i)$ and $\alpha(t) \le x_i(t) \le \alpha(t) + 1$, $t \in [c_{i-1}, c_i]$. Define α_1 as

$$\alpha_1(t) = x_i(t), \quad t \in [c_{i-1}, c_i], \quad i = 1, \ldots, m.$$

A function x_i will be called *an arc* with the end points at c_{i-1} and c_i. Let us show that $\alpha_1 \in A(I, I\!R)$. It is clear that $\alpha_1 \in \mathrm{Lip}(I, I\!R)$. If t_1 and t_2 belong to the same interval $[c_{i-1}, c_i]$, the inequality (4) is evident. Consider the case of $t_1 \in [c_{i-1}, c_i)$, $t_2 \in (c_i, c_{i+1}]$. Then

$$\phi(c_i, \alpha_1(c_i), \alpha'_{1l}(c_i)) - \phi(t_1, \alpha_1(t_1), \alpha'_1(t_1)) = \int_{t_1}^{c_i} f(s, \alpha_1(s), \alpha'_1(s))\, ds, \tag{11}$$

$$\phi(t_2, \alpha_1(t_2), \alpha'_1(t_2)) - \phi(c_i, \alpha_1(c_i), \alpha'_{1r}(c_i)) = \int_{c_i}^{t_2} f(s, \alpha_1(s), \alpha'_1(s))\, ds. \tag{12}$$

The inequalities $\alpha'_{1l}(c_i) \le \alpha'_{1r}(c_i)$ and (11), (12) together imply (4). Proof for other cases can be obtained by induction.

To construct α_2 apply the above arguments to intervals $[c_{i-1}, c_i]$ with $\varepsilon = \frac{1}{2}$. We may assume now that the length of the maximal interval is less than $\frac{1}{2}$. Proceeding in the same manner, construct α_n assuming that $\varepsilon = \frac{1}{n}$ and the length of the maximal interval is less than $\frac{1}{n}$. Evidently the sequence of α_n uniformly converges to α.

Let us show now that if $\alpha'(t)$ exists for some $t \in (a, b)$, which is not a partition point, then $\alpha'_n(t) \to \alpha'(t)$ as $n \to \infty$. For any $\varepsilon \in (0, 1)$ there exists $\delta \in (0, +\infty)$ such that $a < t - \delta < t + \delta < b$, the graph of $\alpha(\tau)$ for any $\tau \in (t - \delta, t + \delta)$ lies between the straight lines defined by $\alpha(t) + (\alpha'(t) + \varepsilon)(\tau - t)$ and $\alpha(t) + (\alpha'(t) - \varepsilon)(\tau - t)$, and

$$\begin{aligned} 2\int_{t-\delta}^{t+\delta} g(s)\, ds < \min\{\phi(t, \alpha(t), \alpha'(t) + 2\varepsilon) &- \phi(t, \alpha(t), \alpha'(t) + \varepsilon), \\ \phi(t, \alpha(t), \alpha'(t) - \varepsilon) &- \phi(t, \alpha(t), \alpha'(t) - 2\varepsilon)\}, \end{aligned}$$

where g (a function from the Carathéodory conditions) depends on $\alpha(t)$ and $\alpha'(t)$ and does not depend on n. Choose n so large that there exists an arc $x : [c, d] \to I\!R$ of the function α_n with the end points $c \in (t - \delta, t)$ and $d \in (t, t + \delta)$ and

$$\begin{aligned} &\phi(\tau_1, \alpha_n(\tau_1), \alpha'(t) + 2\varepsilon) - \phi(\tau_2, \alpha_n(\tau_2), \alpha'(t) + \varepsilon) \\ &\qquad > (\phi(t, \alpha(t), \alpha'(t) + 2\varepsilon) - \phi(t, \alpha(t), \alpha'(t) + \varepsilon))/2, \\ &\phi(\tau_1, \alpha_n(\tau_1), \alpha'(t) - \varepsilon) - \phi(\tau_2, \alpha_n(\tau_2), \alpha'(t) - 2\varepsilon) \\ &\qquad > (\phi(t, \alpha(t), \alpha'(t) - \varepsilon) - \phi(t, \alpha(t), \alpha'(t) - 2\varepsilon))/2 \end{aligned} \tag{13}$$

for $\tau_1, \tau_2 \in [c, d]$. Then

$$\alpha'(t) - 2\varepsilon \leq x'(t) \leq \alpha'(t) + 2\varepsilon. \tag{14}$$

Indeed, if $x'(t) > \alpha'(t) + 2\varepsilon$, then there exist $t_1, t_2 \in (t, d]$ such that $t_1 < t_2$, $\alpha_n'(t_1) = \alpha'(t) + 2\varepsilon$, $\alpha_n'(t_2) = \alpha'(t) + \varepsilon$ and $\alpha'(t) + \varepsilon \leq \alpha_n'(\tau) \leq \alpha'(t) + 2\varepsilon$, $\tau \in (t_1, t_2)$. Hence

$$\begin{aligned} -\int_{t_1}^{t_2} g(s)\, ds &\leq \int_{t_1}^{t_2} f(s, \alpha_n(s), \alpha_n'(s))\, ds \\ &\leq \phi(t_2, \alpha_n(t_2), \alpha'(t) + \varepsilon) - \phi(t_1, \alpha_n(t_1), \alpha'(t) + 2\varepsilon) \\ &< (\phi(t, \alpha(t), \alpha'(t) + \varepsilon) - \phi(t, \alpha(t), \alpha'(t) + 2\varepsilon))/2 \\ &< -\int_{t-\delta}^{t+\delta} g(s)\, ds. \end{aligned}$$

The contradiction obtained proves the estimate (14) from above. The estimate from below can be proved similarly. The estimate (14) implies that $\alpha_n'(t)$ converge to $\alpha'(t)$.

Let $M = \max\{|\alpha(t)| : t \in I\} + 1$, $\alpha \in \mathrm{Lip_L}(I, I\!R)$, $P = [-M, M] \times [-L-1, L+1]$ and

$$2\varepsilon_1 = \min\{\phi(t, \alpha(t), L+1) - \phi(t, \alpha(t), L) : \quad t \in I\},$$
$$2\varepsilon_2 = \min\{\phi(t, \alpha(t), -L) - \phi(t, \alpha(t), -L-1) : \quad t \in I\}.$$

Let g be a summable function from the Carathéodory conditions, which corresponds to the set P. Let n_0 be such that

$$|\int_{t_1}^{t_2} g(t)\, dt| < \min\{\varepsilon_1, \varepsilon_2\}, \quad t_1, t_2 \in I, \quad |t_1 - t_2| < \frac{1}{n_0},$$

$$\phi(t_2, \alpha_n(t_2), L+1) - \phi(t_1, \alpha_n(t_1), L) > \varepsilon_1,\ t_1, t_2 \in I, \quad |t_1 - t_2| < \frac{1}{n_0},\ n > n_0,$$

$$\phi(t_2, \alpha_n(t_2), -L) - \phi(t_1, \alpha_n(t_1), -L-1) > \varepsilon_2,\ t_1, t_2 \in I,\ |t_1 - t_2| < \frac{1}{n_0},\ n > n_0.$$

We show now that $\alpha_n \in \mathrm{Lip_{L+1}}(I, I\!R)$, $n > n_0$. If there exists $\tau \in [c, d]$ for an arc $x : [c, d] \to I\!R$ of α_n such that $x'(\tau) > L+1$, then there exist $t_1, t_2 \in [c, d]$ such that $x'(t_1) = L$, $x'(t_2) = L+1$, $t_2 < t_1$, and $L < x'(t) < L+1$, $t \in (t_2, t_1)$. We have then that

$$\varepsilon_1 < \phi(t_2, x(t_2), L+1) - \phi(t_1, x(t_1), L) \leq \int_{t_1}^{t_2} f(t, x(t), x'(t))dt \leq \int_{t_2}^{t_1} g(t)dt < \varepsilon_1$$

and this proves the estimate $\alpha_n'(t) \leq L+1$, $t \in I$, $n > n_0$. The inequality $-L-1 \leq \alpha_n'(t)$, $t \in I$, $n > n_0$ can be proved similarly.

Passing to the limit in

$$\phi(t_2, \alpha_n(t_2), \alpha_n'(t_2)) - \phi(t_1, \alpha_n(t_1), \alpha_n'(t_1)) \geq \int_{t_1}^{t_2} f(s, \alpha_n(s), \alpha_n'(s))\, ds,$$

one gets the inequality (4) for any points of T_α except possibly the partition points. It follows from the proof of Lemma 2.1 that the first condition in (6) holds if the first condition in (4) is true for any set $T \subset T_\alpha$ of full measure. Hence $\alpha \in A(I, I\!R)$.

Proof for β can be conducted similarly.

Theorem 2.2 *If $\alpha \in A(I, I\!R)$ then α satisfies the one-sided local solvability condition from above. If $\beta \in B(I, I\!R)$ then β satisfies the one-sided local solvability condition from below.*

Proof. Let L be a Lipschitz constant for α. First we prove that there exists $\delta \in (0, +\infty)$ such that for any $c \in [a, b)$ and $d \in (c, b]$ the inequality $d - c < \delta$ implies the existence of a solution $x : [c, d] \to I\!R$ to the Dirichlet problem

$$(\phi(t, x, x'))' = f(t, x, x'),$$

$$x(c) = \alpha(c), \ \ x(d) = \alpha(d), \tag{15}$$

which satisfies the estimate

$$|x'(t)| \le L + 1, \quad t \in [c, d]. \tag{16}$$

Denote

$$X_1[c, d] = \{x \in C^1([c, d], I\!R) : x \text{ satisfies (15) and (16)}\}.$$

By definition, functions from X_1 are uniformly bounded with respect to c and d. The Carathéodory conditions imply that a function $g \in L_1(I, (0, +\infty))$ exists such that $|f(t, x(t), x'(t))| \le g(t)$, $t \in [c, d]$, for any $x \in X_1[c, d]$. Let $G(t) = \int_a^t g(s)\,ds$ and let ω_1 stand for a modulus of continuity of the function G. Choose $\delta > 0$ so that

$$\min\{\phi(t, \alpha(t), L + 1) - \phi(t, \alpha(t), L) : t \in I\} > 2\omega_1(\delta),$$

$$\min\{\phi(t, \alpha(t), -L) - \phi(t, \alpha(t), -L - 1) : t \in I\} > 2\omega_1(\delta).$$

We may suppose (taking δ smaller if needed) that

$$\begin{aligned} \min\{\phi(t_1, x_1, L + 1) \ &- \ \phi(t_2, x_2, L) : t_1, t_2 \in I, |t_2 - t_1| < \delta, |x_i - \alpha(t_i)| \\ &< \ (L + 1)\delta, \quad i = 1, 2\} > \omega_1(\delta), \\ \min\{\phi(t_1, x_1, -L) \ &- \ \phi(t_2, x_2, -L - 1) : t_1, t_2 \in I, |t_2 - t_1| < \delta, \\ & \quad\ |x_i - \alpha(t_i)| < (L + 1)\delta, \quad i = 1, 2\} > \omega_1(\delta). \end{aligned} \tag{17}$$

Introduce $z = \phi(t, x, y)$. By Implicit Function Theorem there exists a continuous function $\psi(t, x, z) = y$. Consider the map $T : X_1[c, d] \to X_1[c, d]$, defined by

$$y(t) = (Tx)(t) = \alpha(c) + \int_c^t \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\,ds + \phi(c, \alpha(c), y'(c)))\,d\tau, \tag{18}$$

where $y'(c)$ is to be determined from the condition

$$\int_c^d \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\,ds + \phi(c, \alpha(c), y'(c)))\,d\tau = \alpha(d) - \alpha(c). \tag{19}$$

Let us show that δ can be chosen so that the expression

$$\psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\,ds + \phi(c, \alpha(c), y'(c))) \tag{20}$$

makes sense, $y'(c) \in [-L-1, L+1]$. Suppose that

$$\int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y'(c)) = \phi(c, \alpha(c), z)$$

and δ is such that $z \in [-L-2, L+2]$. Consider

$$\phi(c, \alpha(c), z) = \phi(\tau, x(\tau), z) + \phi(c, \alpha(c), z) - \phi(\tau, x(\tau), z) = \phi(\tau, x(\tau), u)$$

and choose δ such that $u \in [-L-3, L+3]$. It is clear then, that the expression (20) makes sense. Let us show that for $y'(c) = L+1$ the left side of (19) is greater than $\alpha(d) - \alpha(c)$. Indeed, set

$$\int_c^d \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), L+1))\, d\tau = \Delta.$$

Then there exists $e \in [c, d]$ such that

$$(d-c)\psi(e, x(e), \int_c^e f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), L+1)) = \Delta,$$
$$\int_c^e f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), L+1) = \phi(e, x(e), \Delta/(d-c)).$$

It follows from (17) that $\Delta \,/\, (d-c) > L$. Hence $\Delta > L(d-c) \geq \alpha(d) - \alpha(c)$. Similarly for $y'(c) = -L-1$ the left side of (19) is less than $\alpha(d) - \alpha(c)$. Then one can find $y'(c) \in (-L-1, L+1)$ from (19). Show now that $T : X_1[c, d] \to X_1[c, d]$. Obviously $y(c) = \alpha(c)$ and (18) and (19) together imply that $y(d) = \alpha(d)$. Let us show that $|y'(t)| \leq L+1$, $t \in [c, d]$. Suppose that $\max\{y'(t) : t \in [c, d]\} > L+1$. Then there exist $t_1, t_2 \in [c, d]$ such that $y'(t_1) = L$, $y'(t_2) = L+1$, $t_2 < t_1$, and for $t \in (t_2, t_1)$ the inequality $L \leq y'(t) \leq L+1$ holds. It follows from (18) that

$$\begin{aligned} &y'(t) = \psi(t, x(t), \int_c^t f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y'(c))), \\ &\phi(t, x(t), y'(t)) - \phi(c, \alpha(c), y'(c)) = \int_c^t f(s, x(s), x'(s))\, ds, \\ &\phi(t_2, x(t_2), y'(t_2)) - \phi(t_1, x(t_1), y'(t_1)) = \int_{t_1}^{t_2} f(s, x(s), x'(s))\, ds. \end{aligned} \tag{21}$$

Hence

$$\begin{aligned} \phi(t_2, x(t_2), L+1) - \phi(t_1, x(t_1), L) &= \int_{t_1}^{t_2} f(s, x(s), x'(s))\, ds \\ &\leq |\int_{t_1}^{t_2} g(s)\, ds| \leq \omega_1(\delta), \end{aligned}$$

which contradicts the inequality (17). The case of $\min\{y'(t) : t \in [c, d]\} < -L-1$ can be considered similarly. Hence $y \in X_1[c, d]$. Let $\varepsilon > 0$,

$$\begin{aligned} &M = \max\{|\alpha(t)| : \ t \in I\} + (L+1)(b-a) \\ &\omega_2(\varepsilon) = \max\{\psi(t_2, x_2, y_2) - \psi(t_1, x_1, y_1) : t_1, t_2 \in I, |t_2 - t_1| \leq \varepsilon, \\ &x_1, x_2 \in [-M, M], \ |x_2 - x_1| \leq \varepsilon(L+1), \\ &y_1, y_2 \in [-L-1, L+1], \ |y_2 - y_1| \leq \omega_1(\varepsilon)\}. \end{aligned}$$

It follows from (21) that the modulus of continuity of y' is less or equal than ω_2. Let $X[c,d]$ be a set of $x \in X_1[c,d]$ with the modulus of continuity of x' being less or equal than ω_2. A set $X[c,d]$ is convex and compact with respect to C^1 norm.

Let us show that the map T is continuous. Let a sequence $x_n \in X[c,d]$ $(n = 1,2,\ldots)$ converge to a limit element x in C^1 norm. Evidently $x \in X[c,d]$. Show that a sequence $y_n = Tx_n$ $(n = 1,2,\ldots)$ converges to $y = Tx$ in C^1 norm. It follows from (19) that

$$\begin{aligned} &\int_c^d \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y'(c)))\, d\tau \\ &= \int_c^d \psi(\tau, x_n(\tau), \int_c^\tau f(s, x_n(s), x_n'(s))\, ds + \phi(c, \alpha(c), y_n'(c)))\, d\tau. \end{aligned} \tag{22}$$

Let us prove now that $y_n'(c) \to y'(c)$ as $n \to \infty$. Suppose the contrary is true. Without loss of generality we may assume that $y_n'(c)$ converges to some $y_*' \neq y'(c)$. Passing to a limit in (22) yields

$$\begin{aligned} &\int_c^d \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y'(c)))\, d\tau \\ &= \int_c^d \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y_*'))\, d\tau, \end{aligned}$$

which is impossible if $y'(c) \neq y_*'$. Therefore $y_n'(c) \to y'(c)$ as $n \to \infty$. The relation

$$\begin{aligned} &\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty}(\alpha(c) + \int_c^t \psi(\tau, x_n(\tau), \int_c^\tau f(s, x_n(s), x_n'(s))\, ds \\ &+\phi(c, \alpha(c), y_n'(c)))\, d\tau) = \alpha(c) + \int_c^t \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds \\ &+\phi(c, \alpha(c), y'(c)))\, d\tau = y(t), \quad t \in [c,d] \end{aligned}$$

implies the pointwise convergence of y_n to y. Since y_n $(n = 1,2,\ldots)$ and y satisfy the Lipschitz condition with a constant $L+1$, sequence y_n uniformly converges to y. The relation (21) yields

$$\begin{aligned} &\lim_{n\to\infty} y_n'(t) = \lim_{n\to\infty} \psi(t, x_n(t), \int_c^t f(s, x_n(s), x_n'(s))\, ds + \phi(c, \alpha(c), y_n'(c))) \\ &= \psi(t, x(t), \int_c^t f(s, x(s), x'(s))\, ds + \phi(c, \alpha(c), y'(c))) = y'(t), \quad t \in [c,d]. \end{aligned}$$

Then y_n' converges to y' pointwisely. Since y_n' $(n = 1,2,\ldots)$ and y' have modula of continuity less or equal then ω_2, pointwise convergence implies uniform convergence of y_n' to y'. This means that y_n converges to y in C^1 norm. Convexity and compactness of $X[c,d]$ and continuity of $T : X[c,d] \to X[c,d]$ together imply the existence of a fixed point $x \in X[c,d]$ such that

$$x(t) = \alpha(c) + \int_c^t \psi(\tau, x(\tau), \int_c^\tau f(s, x(s), x'(s))\, ds + \phi(c, x(c), x'(c)))\, d\tau$$

or

$$(\phi(t, x(t), x'(t))' = f(t, x(t), x'(t)), \quad t \in [c,d]. \tag{23}$$

To complete the proof for α, it suffices to show that there exists a solution $x \in X[c,d]$ of the equation (23), which satisfies the inequality $x \geq \alpha$. For this define ϕ_* and f_* as follows

$$\begin{array}{rcll} \phi_*(t,x,y) &=& \phi(t,\alpha(t),y), & x \leq \alpha(t) \\ \phi_*(t,x,y) &=& \phi(t,x,y), & x > \alpha(t) \\ f_*(t,x,x') &=& f(t,\alpha(t),\alpha'(t)), & x \leq \alpha(t) - 2|x'-\alpha'(t)|, \\ f_*(t,x,x') &=& f(t,\alpha(t),x'), & \alpha(t) - |x'-\alpha'(t)| \leq x \leq \alpha(t), \\ f_*(t,x,x') &=& f(t,x,x'), & \alpha(t) \leq x, \end{array}$$

$f_*(t,x,x')$ is linear with respect to x for $\alpha(t) - 2|x'-\alpha'(t)| \leq x \leq \alpha(t) - |x'-\alpha'(t)|$ and t, x' fixed. It follows from the arguments above that a solution $x \in X[c,d]$ of the equation

$$(\phi_*(t,x,x'))' = f_*(t,x,x'), \quad t \in [c,d] \tag{24}$$

exists. Show that $x \geq \alpha$. Suppose this is not the case. Let $t_0 \in (c,d)$ be such that $\max\{\alpha(t) - x(t) : t \in [c,d]\} = \alpha(t_0) - x(t_0)$ and $\alpha(t_0) - x(t_0) > \alpha(t) - x(t)$ for $t \in (t_0, d]$. It follows from the maximum condition that $\alpha'_l(t_0) - x'(t_0) \geq \alpha'_r(t_0) - x'(t_0)$ and Lemma 2.1 implies that $\alpha'_l(t_0) \leq \alpha'_r(t_0)$. Therefore $\alpha'(t_0)$ exists and $\alpha'(t_0) = x'(t_0)$. It follows from the condition (5) that $f_*(s, x(s), x'(s)) = f(s, \alpha(s), \alpha'(s))$ for $s \in (t_0, t_0+\varepsilon)$, if $\varepsilon > 0$ is sufficiently small. Hence for $t \in (t_0, t_0+\varepsilon)$ one has

$$\begin{array}{rcl} \alpha'(t) - x'(t) &\geq& \psi(t,\alpha(t), \displaystyle\int_{t_0}^{t} f(s,\alpha(s),\alpha'(s))\,ds + \phi(t_0,\alpha(t_0),\alpha'(t_0))) \\ &-& \psi_*(t,x(t), \displaystyle\int_{t_0}^{t} f_*(s,x(s),x'(s))\,ds + \phi_*(t_0,x(t_0),x'(t_0))) = 0, \end{array}$$

which is in contradiction with the definition of t_0. Thus $x \geq \alpha$.

Proof for β is analogous.

Remark 1. If $\alpha \in A(I,I\!\!R)$, $\beta \in B(I,I\!\!R)$ and $\alpha \leq \beta$, then in the definition of a local one-sided solvability one might assume in addition that $\alpha(t) \leq x(t) \leq \beta(t)$, $t \in [t_1, t_2]$.

Remark 2. In a similar manner it is possible to prove a local solvability of the Cauchy problem

$$(\phi(t,x,x'))' = f(t,x,x'), \quad x(a) = A, x'(a) = A_1, \quad A, A_1 \in I\!\!R.$$

In the work [6] one can find another definition of α and β. We state it below for convenient reference.

Definition 4. A function $\alpha : I \to I\!\!R$ is called by a lower function of equation (3), if $\alpha \in C(I,I\!\!R) \cap BV(I,I\!\!R)$, $\alpha' \in L_\infty(I,I\!\!R)$, $D_-\alpha(t) \leq D^+\alpha(t)$, $t \in (a,b)$, and if $D_-\alpha(t_0) = D^+\alpha(t_0)$ for some $t_0 \in (a,b)$, then $\varepsilon > 0$ exists such that $\phi(t,\alpha,D^+\alpha) \in BV^+([t_0,t_0+\varepsilon],I\!\!R)$ and the inequality

$$(\phi(t,\alpha(t),D^+\alpha(t)))' \geq f(t,\alpha(t),D^+\alpha(t)) \tag{25}$$

holds a.e. on $[t_0, t_0+\varepsilon]$. An upper function β is defined similarly.

Theorem 2.3 *Definitions 2 and 4 are equivalent.*

Proof. Let $\alpha \in A(I, I\!R)$. We show that α satisfies the definition above. Since α is lipschitzian, the inclusions $\alpha \in C(I, I\!R) \cap BV(I, I\!R)$ and $\alpha' \in L_\infty(I, I\!R)$ follow. By Lemma 2.1 there exist $\alpha_l'(t)$, $t \in (a, b]$ and $\alpha_r'(t)$, $t \in [a, b)$, the inequalities (6) (with respect to α) and $\alpha_l'(t) \le \alpha_r'(t)$, $t \in (a, b)$ hold. The latter one implies that $D_-\alpha(t) \le D^+\alpha(t)$, $t \in (a, b)$. The inclusion $\phi(t, \alpha, D^+\alpha) \in BV^+([t_0, t_0 + \varepsilon], I\!R)$ and (25) together are equivalent to the condition: for any $t_1 \in [t_0, t_0 + \varepsilon)$ and $t_2 \in (t_1, t_0 + \varepsilon]$ the inequality

$$\phi(t_2, \alpha(t_2), D^+\alpha(t_2)) - \phi(t_1, \alpha(t_1), D^+\alpha(t_1)) \ge \int_{t_1}^{t_2} f(s, \alpha(s), D^+\alpha(s))\, ds \qquad (26)$$

holds. In our case the inequality (26) is equivalent to

$$\phi(t_2, \alpha(t_2), \alpha_r'(t_2)) - \phi(t_1, \alpha(t_1), \alpha_r'(t_1)) \ge \int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))\, ds,$$

which follows from (6) and $\alpha_l'(t_2) \le \alpha_r'(t_2)$.

We will show now that Definition 4 implies Definition 2 on the interval $[t_0, t_0 + \varepsilon]$, where ε is sufficiently small. If there exists $\alpha'(t_1)$ and $\alpha'(t_2)$ then (26) implies (4). It remains to prove that α is lipschitzian on $[t_0, t_0 + \varepsilon]$. Let $|\alpha'(t)| \le L$ for a.e. $t \in I$. We prove that the limit

$$\lim_{t \to t_0+} D^+\alpha(t) =: \alpha_r(t_0) \qquad (27)$$

exists. Suppose this is not the case and

$$\liminf_{t \to t_0+} D^+\alpha(t) < c_1 < c_2 < \limsup_{t \to t_0+} D^+\alpha(t). \qquad (28)$$

Pick $t_1 \in (t_0, t_0 + \varepsilon)$ and $t_2 \in (t_1, t_0 + \varepsilon)$ such that $D^+\alpha(t_1) > c_2$, $D^+\alpha(t_2) < c_1$ and

$$\begin{aligned} \frac{\phi(t_0, \alpha(t_0), c_1) - \phi(t_0, \alpha(t_0), c_2)}{2} &< \int_{t_1}^{t_2} f(s, \alpha(s), \alpha'(s))\, ds \\ &\le \phi(t_2, \alpha(t_2), D^+\alpha(t_2)) - \phi(t_1, \alpha(t_1), D^+\alpha(t_1)) \\ &< \phi(t_2, \alpha(t_2), c_1) - \phi(t_1, \alpha(t_1), c_2) < (\phi(t_0, \alpha(t_0), c_1) - \phi(t_0, \alpha(t_0), c_2))/2. \end{aligned}$$

The contradiction obtained proves the existence of a limit in (27). It is clear that $|\alpha_r(t_0)| \le L$. The inequality $|D^+\alpha(t)| < L + 1$, $t \in [t_0, t_0 + \varepsilon)$ follows from (27) for ε sufficiently small. Hence $D^+(\alpha(t) + (L + 1)t) > 0$, $t \in [t_0, t_0 + \varepsilon)$. The function $\alpha(t) + (L + 1)t$ is increasing ([4, p. 189]). Analogously the function $\alpha(t) - (L + 1)t$ is decreasing. Therefore α satisfies the Lipschitz condition on $[t_0, t_0 + \varepsilon]$ with a constant $L + 1$. It follows from Lemma 2.1 that $\alpha_r'(t_0)$ exists and $\alpha_r'(t_0) = \alpha_r(t_0)$.

Thus the derivative $\alpha_r'(t_0)$ exists at any point $t_0 \in (a, b)$, where $D_-\alpha(t_0) = D^+\alpha(t_0)$, and $|\alpha_r'(t_0)| \le L$. There is at most countable set of points such that $D_-\alpha(t) < D^+\alpha(t)$ ([4, p. 63]). This means that the above arguments can be applied to prove that α is lipschitzian function with a constant L on a whole interval I.

The proof of the local solvability from above for Definition 4 is analogous to the proof of Theorem 2.2. By Theorem 2.1 then $\alpha \in A(I, I\!R)$.

Lemma 2.3 *If* $\alpha \in A(I, I\!R)$ *and* $\beta \in B(I, I\!R)$, $\alpha \le \beta$,
$|f(t,x,x')| \le g(t)$ *for a.e.* $t \in I$, $\alpha \le x \le \beta$, $x' \in I\!R$, *where* $g \in L_1(I, I\!R)$,
$\inf\{\phi(t_1, x_1, L) - \phi(t_2, x_2, \frac{\beta(b)-\alpha(a)}{b-a}) : t_1, t_2 \in I,\ x_1, x_2 \in I\!R\} > G$,
$\inf\{\phi(t_1, x_1, \frac{\beta(a)-\alpha(b)}{b-a}) - \phi(t_2, x_2, -L) : t_1, t_2 \in I,\ x_1, x_2 \in I\!R\} > G$,
$\inf\{\phi(t_1, x_1, L_1) - \phi(t_2, x_2, L) : t_1, t_2 \in I,\ x_1, x_2 \in I\!R\} > G$,
$\inf\{\phi(t_1, x_1, -L) - \phi(t_2, x_2, -L_1) : t_1, t_2 \in I,\ x_1, x_2 \in I\!R\} > G$,
where $G = \int_a^b g(t)\, dt$, $L, L_1 \in (0, +\infty)$,
then for any $A \in [\alpha(a), \beta(a)]$ *and* $B \in [\alpha(b), \beta(b)]$ *there exists a solution to the Dirichlet problem*

$$(\phi(t,x,x'))' = f(t,x,x'), \quad x(a) = A, \quad x(b) = B, \quad \alpha \le x \le \beta. \tag{29}$$

Proof. First, let functions $\phi_*(t,x,x')$ and $f_*(t,x,x')$ be defined analogously as in the proof of Theorem 2.2. Consider equation

$$(\phi_*(t,x(t),x'(t)))' = f_*(t,x(t),x'(t)), \qquad x(a) = A, \quad x(b) = B. \tag{30}$$

It can be shown likely as in the proof of Theorem 2.2 that a solution x of the problem (30) (if any) satisfies the estimate $\alpha \le x \le \beta$.

We will show now that the problem (30) has a solution. Let $X(A,B)$ stand for a set of functions $x \in C^1(I, I\!R)$, which satisfy the following conditions: $x \in \mathrm{Lip}_L(I, I\!R)$, $x(a) = A$, $x(b) = B$ and the modulus of continuity of x' is less or equal to ω_2. Set $z = \phi_*(t,x,y)$ and $y = \psi_*(t,x,z)$. Consider mapping $T : X(A,B) \to X(A,B)$, which is defined by

$$y(t) = (Tx)(t) = A + \int_a^t \psi_*(\tau, x(\tau), \int_a^\tau f_*(s, x(s), x'(s))\, ds + \phi_*(a, A, y'(a)))\, d\tau, \tag{31}$$

where $y'(a)$ is defined by

$$\int_a^b \psi_*(\tau, x(\tau), \int_a^\tau f_*(s, x(s), x'(s))\, ds + \phi_*(a, A, y'(a)))\, d\tau = B - A. \tag{32}$$

Evidently $y(a) = A$, and the equality $y(b) = B$ follows from (31) and (32). Arguing as in the proof of Theorem 2.2 one can show that the relation

$$\psi_*(\tau, x(\tau), \int_a^\tau f_*(s, x(s), x'(s))\, ds + \phi_*(a, A, y'(a))), \quad y'(a) \in [-L, L]$$

makes sense, $y \in X(A,B)$, mapping T is continuous and there exists a fixed point $x \in X(A,B)$ such that

$$x(t) = A + \int_a^t \psi_*(\tau, x(\tau), \int_a^\tau f_*(s, x(s), x'(s))\, ds + \phi_*(a, A, x'(a)))\, d\tau$$

or

$$(\phi_*(t,x(t),x'(t)))' = f_*(t,x(t),x'(t)), \quad t \in I. \tag{33}$$

Since equations (33) and (29) are equivalent for $\alpha \le x \le \beta$, the assertion of the lemma follows.

Definition 5 ([17]). Let $\alpha, \beta : [a,b] \to I\!R$ and $\alpha \le \beta$. We say that *the Schrader condition* is fulfilled for equation (3) between α and β, if for any $c \in [a,b), d \in (c,b]$ and for any solution $x : (c,d) \to I\!R$ of the equation (3) the estimate $\alpha(t) \le x(t) \le \beta(t), \quad t \in (c,d)$ implies that $\sup\{|x'(t)| : t \in (c,d)\} < \infty$.

Theorem 2.4 *Suppose that* $\alpha \in A(I, I\!R), \quad \beta \in B(I, I\!R), \quad \alpha \le \beta, \quad A \in [\alpha(a), \beta(a)], \quad B \in [\alpha(b), \beta(b)]$ *and the Schrader condition is fulfilled for equation (3) between* α *and* β.

Then the Dirichlet problem (29) has minimal and maximal solutions.

Proof. Let $N > 0$ be such that $\max\{|\alpha'(t)|, |\beta'(t)|\} < N$ a. e. on the interval I. Consider the Dirichlet problem

$$\begin{aligned}(\phi(t, \delta(\alpha(t), x(t), \beta(t)), x'(t)) + \max\{0, x'(t) - N\} + \min\{0, x'(t) + N\})' \\ = f(t, \delta(\alpha(t), x(t), \beta(t)), \delta(-N, x'(t), N)), \\ x(a) = A, \quad x(b) = B, \quad \alpha \le x \le \beta,\end{aligned} \tag{34}$$

where

$$\delta(u, z, v) = \begin{cases} v, & \text{if } z > v \\ z, & \text{if } u \le z \le v \\ u, & \text{if } z < u. \end{cases}$$

By Lemma 2.3 the Dirichlet problem (34) has a solution x_N. If $|x_N'(t)| < N, \ t \in I$, then x_N is a solution of the Dirichlet problem (29). Let us show that $|x_N'(t)| < N, \ t \in I$, for some N. Suppose the contrary is true. Consider the case when for solutions x_n there exist intervals $[a_n, b_n] \subset [a,b]$ such that

$$\begin{aligned}\max\{|x_n'(t)| : t \in [a_n, b_n]\} = x_n'(b_n) = n, \\ \min\{|x_n'(t)| : t \in [a_n, b_n]\} = x_n'(a_n) = \tfrac{B-A}{b-a}.\end{aligned}$$

Evidently x_n satisfies equation (3) on the interval $[a_n, b_n]$. Passing to a subsequence if needed, one can show that $\{x_n\}$ converges to a solution $x : (c,d) \to I\!R$ of equation (3), for which $\alpha(t) \le x(t) \le \beta(t), \quad t \in (c,d)$ and $\sup\{|x'(t)| : t \in (c,d)\} = \infty$, and this is in contradiction with the Schrader condition. Other cases can be considered in a similar manner.

Let us show that there exists maximal solution of the Dirichlet problem (29). If there are finite number of solutions to the problem (29), then the existence of maximal solution follows from Lemma 2.2. Consider the case of infinitely many solutions. Let $D(A,B)$ stand for a set of solutions of the problem (29). Denote $\alpha_*(t) = \sup\{x(t) : x \in D(A,B)\}, \ t \in I$ and let r_i $(i = 1, 2, \ldots)$ stand for all rational points of the interval I. It follows from the compactness of the set $D(A,B)$ that there exists $x_1 \in D(A,B)$ such that $x_1(r_1) = \alpha_*(r_1)$. By Lemma 2.2 there exists $x_2 \in D(A,B)$ such that $x_2 \ge x_1$ and $x_2(r_2) = \alpha_*(r_2)$. Proceeding in the same manner, one obtains a monotone sequence x_i $(i = 1, 2, \ldots)$, which converges to α_*. Hence $\alpha_* \in D(A,B)$. Proof in the case of minimal solution is similar.

Remark 3. Theorem 2.4 implies the global solvability of the Dirichlet problem. If $\alpha \in A(I, I\!R)$, $\beta \in B(I, I\!R)$, $\alpha \le \beta$ and the Schrader condition for equation (3) between α and β holds, then for any $c \in [a,b)$, $d \in (c,b]$, $C \in [\alpha(c), \beta(c)]$ and $D \in [\alpha(d), \beta(d)]$ there exists a solution $x : [c,d] \to I\!R$ of equation (3) with the properties that $x(c) = C, x(d) = D$ and $\alpha(t) \le x(t) \le \beta(t), \ t \in [c,d]$.

Remark 4. Let $\alpha, \beta \in \mathrm{Lip}(I, I\!R)$ and $\alpha \le \beta$. If the global solvability takes place between α and β, then by Theorem 2.1 $\alpha \in A(I, I\!R)$ and $\beta \in B(I, I\!R)$.

3 Bernstein - Nagumo type conditions

The role which play the Bernstein - Nagumo type conditions in the theory of the second order differential equation

$$x'' = f(t, x, x'), \quad t \in I = [a, b] \tag{35}$$

is well known. For discussion one may consult [1, Ch. I, §§1.4, 1.12], [10, Ch. XII,Lemma 5.1], [13, Ch. III, §2].

The first condition of this type was obtained by S. Bernstein ([2], [3]), who studied the Euler - Lagrange equation in the classical calculus of variations. He proved that if (i) $\frac{\partial f}{\partial x} \geq constant > 0$ and (ii) $|f(t, x, y)| \leq C_1 + C_2 y^2$, then there exists a solution $x(t)$ of equation (35), which satisfies the Dirichlet boundary conditions $x(a) = A, \quad x(b) = B$ for any $A, B \in I\!R$. The analysis of conditions (i) and (ii) (by the Bernstein condition usually is meant the latter one) reveals that the condition (i) implies the existence of α and β, described in the preceding section (any positive constant M, greater than $|A|$ and $|B|$, may be chosen as β, $\alpha = -\beta$.). The condition (ii) itself guarantees the existence of a constant $N(M)$ such that any solution $x : I \to I\!R$ of (35), which satisfies the estimate

$$|x(t)| \leq M, \quad t \in I$$

satisfies also the estimate

$$|x'(t)| \leq N, \quad t \in I \tag{36}$$

and N is a function of M only.

Later M. Nagumo obtained ([15]) a generalization of the Bernstein condition. Suppose that a positive valued continuous function ψ exists such that

$$|f(t, x, y)| \leq \psi(|y|), \quad t \in I, \ |x| \leq M, \ |y| \text{ large}$$
$$\int^{\infty} \frac{s\,ds}{\psi(s)} = +\infty. \tag{37}$$

Then there exists $N(M)$ as above. The Bernstein function $C_1 + C_2 y^2$ is good for any M.

Remark 5. Both the Bernstein and Nagumo conditions guarantee that no bounded solution of (35) has an unbounded derivative $x'(t)$, going to ∞ in a finite time, thus implying the Schrader condition ([17]) described in the preceding section.

The Nagumo condition may be weakened if a priori bound on $x(t)$ is known, that is a constant M is known. Then the integral in (37) can be convergent. It is important only that

$$\int_{\frac{2M}{b-a}}^{N} \frac{s\,ds}{\psi(s)} > 2M. \tag{38}$$

Refinements and further generalizations can be found in [11], [8], [16]. For discussion about Bernstein - Nagumo type conditions one may consult also the books [1], [10], [13].

In addition to conditions imposed on ϕ and f in the previous section we assume in the sequel that given α and β such that $\alpha \leq \beta$

$$\lim_{x' \to -\infty} \phi(t, x, x') = -\infty, \quad \lim_{x' \to +\infty} \phi(t, x, x') = +\infty \tag{39}$$

and the limits are uniform in (t,x), $a \le t \le b$, $\alpha \le x \le \beta$.

The next statement is a generalization of the result by I. Kiguradze and N. Lezhava [12].

Theorem 3.1 *Assume that:*

1) the inequality

$$|f(t,x,y)| \le \left(g_0(t) + \sum_{i=1}^{n} g_i(t)\, h_i(x)\, |y|^{\frac{1}{q_i}} \right) \cdot \omega(|\phi(t,x,y)|) \tag{40}$$

holds for $t \in I$, $\alpha(t) \le x \le \beta(t)$, $|y| \ge \lambda := \frac{\beta^* - \alpha_*}{b-a}$, *where* $\omega : (0,+\infty) \to (0,+\infty)$ *is a continuous function,* $g_0 \in L^1(I)$, $g_i \in L^{p_i}(I)$, $h_i \in C([\alpha_*, \beta^*])$ *are non-negative valued functions,* $\alpha_* = \min_I \alpha(t)$, $\beta^* = \max_I \beta(t)$, $q_i \ge 1$ *and* $\frac{1}{p_i} + \frac{1}{q_i} = 1$ *(* $i = 1,2,\ldots,n;$*)*

2) the inequality

$$\int_{\Lambda}^{+\infty} \frac{ds}{\omega(s)} > \|g_0\|_{L^1(I)} + \sum_{i=1}^{n} \|g_i\|_{L^{p_i}(I)} \cdot \|h_i\|_{L^{q_i}([\alpha_*, \beta^*])} =: \mu \tag{41}$$

holds, where

$$\Lambda := \max\{|\phi(t,x,y)| : a \le t \le b,\ \alpha \le x \le \beta, |y| \le \lambda\}.$$

Then a constant $N > 0$ *exists such that for any solution* $x : I \to I\!R$ *of the equation (3), which satisfies* $\alpha \le x \le \beta$*, the estimate (36) is valid.*

Proof. Given $N > 0$, set

$$K(N) := \min\{|\phi(t,x,y)| : a \le t \le b,\ \alpha \le x \le \beta, |y| \ge N\}.$$

Choose N such that $\int_{\Lambda}^{K(N)} \frac{ds}{\omega(s)} > \mu$. This is possible in view of (39) and (41). We will show now that the estimate (36) holds. Suppose this is not the case. Then $|x'(t_1)| = N$ at some point t_1. Notice that $|x'(t_0)| = \lambda$ at some point $t_0 \in I$. The following four cases are possible:

$$\begin{aligned} x'(t_0) &= \lambda, & x'(t_1) &= N, & t_1 &> t_0; \\ x'(t_0) &= \lambda, & x'(t_1) &= N, & t_1 &< t_0; \\ x'(t_0) &= -\lambda, & x'(t_1) &= -N, & t_1 &> t_0; \\ x'(t_0) &= -\lambda, & x'(t_1) &= -N, & t_1 &< t_0. \end{aligned}$$

Consider the first one. Evidently $\Phi(t_0) \le \Lambda$ and $\Phi(t_1) \ge K(N)$, where $\Phi(t) := \phi(t, x(t), x'(t))$. Then there exist $\xi \in [t_0, t_1)$ and $\eta \in (\xi, t_1]$ such that

$$\Phi(\xi) = \Lambda, \quad \Phi(\eta) = K(N), \quad \Lambda \le \Phi(t) \le K(N) \quad \forall t \in (\xi, \eta).$$

Using (40), one has for $t \in (\xi, \eta)$

$$\Phi'(t) = f(t, x(t), x'(t)) \le \left(g_0(t) + \sum_{i=1}^{n} g_i(t)\, h_i(x(t))\, x'(t)^{\frac{1}{q_i}} \right) \cdot \omega(\Phi(t)). \tag{42}$$

Dividing both sides of (42) by $\omega(\Phi(t))$, integrating over the interval $[\xi,\eta]$ and using the Hölder inequality, one gets

$$\begin{aligned}\int_{\xi}^{\eta}\frac{\Phi'(t)\,dt}{\omega(\Phi(t))} &= \int_{\Lambda}^{K(N)}\frac{ds}{\omega(s)} \le \int_{\xi}^{\eta} g_0(t)\,dt + \int_{\xi}^{\eta} g_i(t)h_i(x(t))x'(t)^{\frac{1}{q_i}}\,dt \\ &\le \|g_0\|_{L^1[\xi,\eta]} + \|g_i\|_{L^{p_i}[\xi,\eta]}\cdot\left(\int_{\xi}^{\eta} h_i^{q_i}(x(t))\,x'(t)\,dt\right)^{\frac{1}{q_i}} \\ &\le \|g_0\|_{L^1(I)} + \|g_i\|_{L^{p_i}(I)}\cdot\left(\int_{\xi}^{\eta} h_i^{q_i}(x(t))\,dx(t)\right)^{\frac{1}{q_i}} \\ &\le \|g_0\|_{L^1(I)} + \|g_i\|_{L^{p_i}(I)}\cdot\left(\int_{\alpha_*}^{\beta^*} h_i^{q_i}(s)\,ds\right)^{\frac{1}{q_i}} \\ &= \|g_0\|_{L^1(I)} + \|g_i\|_{L^{p_i}(I)}\cdot\|h_i\|_{L^{q_i}[\alpha_*,\beta^*]}.\end{aligned} \tag{43}$$

The above estimate is in contradiction with (41). Other cases can be treated similarly.

Hence the proof.

Corollary 3.1 *Suppose that*

$$|f(t,x,y)| \le c\cdot|y|\cdot|\phi(t,x,y)| \tag{44}$$

for $t\in I$, $\alpha\le x\le\beta$, $|y|\ge\lambda$, *where* λ *is as above and* c *is a positive constant.*

Then conclusion of Theorem 3.1 is true.

Proof. Choose $g_0=0$, $n=1$, $g_1=c$, $h_1=1$, $q_1=1$ and $\omega(s)=s$.

Remark 6. If $\Phi(y)\equiv y$, then (44) reduces to $|f(t,x,y)|\le c\,y^2$ and this corresponds to the Bernstein condition for equation (35).

Corollary 3.2 *Suppose that*

$$|f(t,x,y)| \le \psi(t)\cdot|y|\cdot\omega(|\phi(t,x,y)|) \tag{45}$$

for $t\in I$, $\alpha\le x\le\beta$, $|y|\ge\lambda$, *where* $\psi\in L^\infty(I,I\!R^+)$, ω *and* λ *are as above and*

$$\int^{\infty}\frac{ds}{\omega(s)} = +\infty.$$

Then conclusion of Theorem 3.1 is true.

Proof. Choose $g_0=0$, $n=1$, $g_1=\psi$, $h_1=1$, $q_1=1$.

Remark 7. If $\Phi(y)\equiv y$ and $\psi\equiv 1$, then (45) reduces to $|f(t,x,y)|\le|y|\,\omega(|y|)$ and we arrive to the Nagumo condition (37) for equation (35) with a Nagumo function ϕ defined by $\phi(s)=s\,\omega(s)$.

Corollary 3.3 *Suppose that* $\phi = \phi(x')$ *in (3) and*

$$|f(t,x,y)| \leq \psi(t) \cdot |y|^{\frac{1}{q}} \cdot \omega(|\phi(y)|) \tag{46}$$

for $t \in I$, $\alpha \leq x \leq \beta$, $|y| \geq \lambda$, *where* $\psi \in L^p(I, I\!R^+)$, ω *and* λ *are as above,* $q \geq 1$, *and*

$$\int_{\Lambda}^{\infty} \frac{ds}{\omega(s)} > \|\psi\|_{L^p(I)} \cdot (\beta^* - \alpha_*)^{\frac{1}{q}},$$

where Λ, β^*, α_* *are as in conditions of Theorem 3.1.*

Then conclusion of Theorem 3.1 is true.

Proof. Choose $g_0 = 0$, $n = 1$, $g_1 = \psi$, $h_1 = 1$.

Remark 8. This condition corresponds to the Nagumo conditions for ϕ-laplacian equations as given in [6] (Definition 1) and [5] for the case $q \equiv 1$ (Definition 2.4). The function $s^{\frac{1}{q}} \cdot \omega(\phi(s))$ serves as $k(s)$ in [6] and $\Theta(s)$ in [5].

Corollary 3.4 *Suppose that*

$$|f(t,x,y)| \leq (g(t) + h(x)|y|) \cdot \omega(|\phi(t,x,y)|) \tag{47}$$

for $t \in I$, $\alpha \leq x \leq \beta$, $|y| \geq \lambda$, *where* $g \in L_1([a,b])$ *and* $h \in C([\alpha_*, \beta^*])$ *are non-negative valued functions,* ω *and* λ *are as in conditions of Theorem 3.1 and*

$$\int_{\Lambda}^{+\infty} \frac{ds}{\omega(s)} > \|g\|_{L^1(I)} + \|h\|_{L^1([\alpha_*, \beta^*])}.$$

Then conclusion of Theorem 3.1 is true.

Proof. Choose $g_0 = g$, $n = 1$, $g_1 = 1$, $h_1 = h$, $q_i = 1$ and apply Theorem 3.1.

Remark 9. This result is a generalization of the one by H. Epheser [8].

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А.Я. Лепин, Л.А. Лепин, Ф.Ж. Садырбаев. Метод нижних и верхних функций для одномерного уравнения с ϕ-Лапласианом.

Аннотация. Рассматривается уравнение $(\phi(t, x, x'))' = f(t, x, x')$ с краевыми условиями $x(a) = A$, $x(b) = B$. Верхние и нижние функции удовлетворяют весьма общим определениям. Показано, что данные определения являются в некотором смысле наилучшими. Во второй части работы получены условия типа Нагумо - Бернштейна для рассматриваемой задачи.

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A.Lepins, L.Lepins, F. Sadirbajevs. Augšējo un apakšējo funkciju metode vienu dimensiju ϕ-laplasian diferenciālvienādojumam.

Anotācija. Tiek apskatīts diferenciālvienādojums $(\phi(t,x,x'))' = f(t,x,x')$ ar robežnosacījumiem $x(a) = A$, $x(b) = B$. Augšējas un apakšējas funkcijas ir ieviestas ar samēra vispārīgu definīciju. Tiek paradīts ka šī definīcija ir zināma nozīme vislabāka. Raksta otrajā daļā tiek iegūti Bernšteina - Nagumo tipa nosacījumi apskatāmam gadījumam.

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